The Quaternionic Exponential (and beyond)

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Motivation

I felt the need to take a closer look at quaternions when, some time back, I was looking for new applications to Harthong-Reeb circles (on which I was working at the time), and came across [D. Pletincks (1989)]. That paper, on one hand, did indicate one potential application for that method, but, on the other hand, alluded to some odd constructions involving quaternions, the validity of which was propitiously left in the shadows. The present text is therefore a compilation of many well-known but apparently scattered results about quaternions (and related entities), as well as some new developments, notably the explicit formula for the quaternionic exponential (and friends). Incidentally, these results enables one to solve the problem found in [D. Pletincks (1989)], but without the unsalvageable constructions.

Chapter 1 Quaternions redux

1- What to find here

This chapter only contains a quick-and-dirty (but sufficient for most uses) presentation of the quaternions, along with their most classical properties, inspired very largely by [D. Leborgne (1982)], [J. Lelong-Ferrand, J.M. Arnaudiès (1978)] and [M. Berger (1990)]. This approach, however, obscures the deep relationship which links the quaternions, the complex and real numbers and more exotic things known as octonions; this relationship will be the thrust of the next chapter.

It should be said that other important uses of quaternions exist ([K. Gürlebeck, W. Spössig (1989)],...), but that they will not be touched upon here. As well, quaternionic analysis ([A. Sudbery (1979)]) and geometry ([S. Salamon (1982)]), though perhaps not as vibrant as their complex counterparts, do keep evolving; though these usually involve fairly sophisticated mathematical machinery, very nice results can also be had with very elementary ones ([P. de Casteljau (1987)],...). All are beyond the scope of this article, however.

2- The nature of the Beast

Let $\mathbf{H}=\mathbf{R}^4$ with the usual four-dimensional vector space structure over $\mathbf{R}$. We define $e=(1,0,0,0)$, $i=(0,1,0,0)$, $j=(0,0,1,0)$ and $k=(0,0,0,1)$.

The first important thing we need is a multiplication, denoted $*$, which we *define* to be a (non-commutative) $\mathbf{R}$-bilinear operation on $\mathbf{H}$ such that $i*i=j*j=k*k=-e$, $i*j=-(j*i)=k$, $j*k=-(k*j)=i$ and $k*i=-(i*k)=j$.

The second important thing we need is the conjugation on $\mathbf{H}$ (and we will usually denote by $\bar{q}$ the conjugate of q) which we define by $(\alpha,\beta,\gamma,\delta)\mapsto(\alpha,-\beta,-\gamma,-\delta)$. Important properties are that $\overline{q*q'}=\bar{q}'*\bar{q}$, that $\bar{e}=e$, that $q*\bar{q}=\bar{q}*q\in\mathbf{R}\cdot e$ and that $q+\bar{q}\in\mathbf{R}\cdot e$. Actually $q*\bar{q}=0$ if and only if $q=0$, as is easily seen.

A straightforward verification then shows that $(\mathbf{H},+,*,\cdot)$ is an effectively non-commutative, but associative, $\mathbf{R}$-algebra with unit e, and that $\left[\mathbf{R}\to\mathbf{H},\,x\mapsto(x,0,0,0)\right]$ and $\left[\mathbf{C}\to\mathbf{H},\,z\mapsto\left(\mathrm{Re}(z),\mathrm{Im}(z),0,0\right)\right]$ are algebra homomorphisms, bijective from their sources onto their images. The image of the conjugate of a complex number is also seen to be

the conjugate (in $\mathbf{H}$) of the image of that complex, by the above function. We will therefore assimilate $\mathbf{H}$ to a superset of (both) $\mathbf{R}$ and $\mathbf{C}$, and identify e with 1 and i with its complex counterpart. We see at once that the operations we have defined on $\mathbf{H}$ extend their counterparts on $\mathbf{C}$ and $\mathbf{R}$. The multiplication can then be memorized thru the well-known formula:

$$i*i = j*j = k*k = i*j*k = -1$$

It is important to notice that given any quaternion q and any *real* number x, we always have $q*x = x*q = x\cdot q$.

We will usually write a quaternion under the form $q = \alpha + \beta i + \gamma j + \delta k$ with α, β, γ and δ reals, omitting the "$\cdot$" when multiplying a quaternion by a real number (as *per* the vector space structure). We will also omit the "$*$" when multiplying a quaternion by a real number, from the left as well as from the right. When no confusion may arise, we will do away entirely with the "$*$".

With the above notations, the conjugate of $q = \alpha + \beta i + \gamma j + \delta k$ will then simply be $\bar{q} = \alpha - \beta i - \gamma j - \delta k$.

Looking at $\mathbf{H}$ as a 4-dimensional $\mathbf{R}$-vector space, it is easy to see the usual euclidian scalar product is equal to the following:

$$\begin{aligned}(p|q) &= (p+q)\overline{(p+q)} - p\bar{p} - q\bar{q}\\ &= \frac{1}{2}(p\bar{q} + q\bar{p})\\ &= \frac{1}{2}(\bar{p}q + \bar{q}p)\end{aligned}$$

All the same, the usual euclidian norm on $\mathbf{R}^4$, coincides with $\left[q \mapsto \|q\| = \sqrt{q*\bar{q}}\right]$, and of course $(q|q) = \|q\|^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$. Note that, if $q \neq 0$ then $q^{-1} = (q*\bar{q})^{-1}\bar{q} = \bar{q}(q*\bar{q})^{-1}$. For the quaternions, we will also use a notation compatible with real and complex numbers and define $|q|$ as $\|q\|$ (of course, if q is actually complex, $|q|$ has exactly the value of the modulus of q).

It is important to remember that $(\mathbf{H}, +, *, \cdot, |\ |)$ is a Banach $\mathbf{R}$-algebra. The norm is better than what we might expect, though, as we have $|p*q| = |p||q|$ instead of just $|p*q| \leq |p||q|$.

We will call the real and unreal parts of quaternion, respectively, $\mathrm{Re}(q) = \frac{1}{2}(\bar{q} + q)$ and $\mathrm{Ur}(q) = \frac{1}{2}(\bar{q} - q)$. We will say that a quaternion is pure if its real part is zero. For a complex number, the quaternionic real part is what is already known as the complex real part, and the unreal part is just the imaginary part multiplied by i.

3- Quaternions' kin

As we have just seen, quaternions are related to both real numbers and complex numbers. As we shall see in some details in the next chapter, quaternions are actually part of an infinite family of sets[1] which we will call the Cayley ladder, some of which we will introduce here as we will have some need of them for our purposes.

First relative in that family, beyond the quaternions, are the octonions. We denote by $\mathbf{O}$ the set $\mathbf{R}^8$, with its usual vector space structure on $\mathbf{R}$, we identify $1=(1,0,0,0,0,0,0,0)$, $i=(0,1,0,0,0,0,0,0)$, $j=(0,0,1,0,0,0,0,0)$ and $k=(0,0,0,1,0,0,0,0)$ and we define $e'=(0,0,0,0,1,0,0,0)$, $i'=(0,0,0,0,0,1,0,0)$, $j'=(0,0,0,0,0,0,1,0)$ and $k'=(0,0,0,0,0,0,0,1)$. We now consider $\mathbf{O}$ to be a super-set of $\mathbf{H}$. We can now define a multiplication on $\mathbf{O}$ by the following table (the value at line n and column m is the product of the element in the left column by the element in the top row; for instance $i*i'=-e'$):

	1	i	j	k	e'	i'	j'	k'
1	1	i	j	k	e'	i'	j'	k'
i	i	-1	k	$-j$	i'	$-e'$	$-k'$	j'
j	j	$-k$	-1	i	j'	k'	$-e'$	$-i'$
k	k	j	$-i$	-1	k'	$-j'$	i'	$-e'$
e'	e'	$-i'$	$-j'$	$-k'$	-1	i	j	k
i'	i'	e'	$-k'$	j'	$-i$	-1	$-k$	j
j'	j'	k'	e'	$-i'$	$-j$	k	-1	$-i$
k'	k'	$-j'$	i'	e'	$-k$	$-j$	i	-1

Other presentations, perhaps more useful, exist ([G. Dixon]). This multiplication still has a unit (1), but is no longer associative (for instance $i'*(e'*j)=+k\neq -k=(i'*e')*j$). Real numbers still commute with every octonion. We define a conjugation by $\overline{\alpha+\beta i+\gamma j+\delta k+\varepsilon e'+\zeta i'+\eta j'+\theta k'}=\alpha-\beta i-\gamma j-\delta k-\varepsilon e'-\zeta i'-\eta j'-\theta k'$, a scalar product and a norm which, as with the quaternions turn out to be exactly the euclidian scalar product and euclidian norm on $\mathbf{R}^8$. Again, we have just extended the quaternionic operations. As with complex numbers and quaternions, we have $|o*o'|=|o||o'|$ for any two octonions o and o', and an octonion o is invertible if and only if it is non-zero, and then $o^{-1}=\frac{1}{|o|^2}\bar{o}$.

Beyond even the octonions, we find $\mathbf{R}^{16}$, which appears not to have any agreed-upon name. We shall here call them hexadecimalions, and denote the set by $\mathbf{X}$ (after the C/C++ notation...). We have the usual vector space structure on $\mathbf{R}$, we identify $1,\ldots,k'$ with $(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),\ldots,(0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0)$ respectively, and define $e'',i'',j'',k'',e''',i''',j''',k'''$ as $(0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0),\ldots,(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)$ respectively. We define a multiplication on $\mathbf{X}$ as explicited in the next chapter, for which 1 is still a unit and for which reals commute with every hexadecimalion. We

[1] Actually, several families, but we will focus on just one here; for others, see [G. Dixon (1994)].

define as well a conjugation, a scalar product and a norm (for details, see next chapter), which once again coincide with the euclidian scalar product and euclidian norm on $\mathbf{R}^{16}$. These all extend the octonionic case. However, the product has even fewer properties than in the octonionic case (the algebra is no longer even alternative[2], as for instance $(i+e''')*((i+e''')*j)=-2j+2k\neq -2j=((i+e''')*(i+e'''))*j$), and the norm is not even an algebra norm any longer, as for instance $\|(i+j'')*(e'+k''')\|^2=8>4=\|i+j''\|^2\,\|e'+k'''\|^2$.

4- Quaternions and rotations

It is pleasant to think that perhaps the relationship between quaternions and rotations has been stumbled upon while running a check-list of classical constructs on the then-newly discovered quaternions. At any rate, the easiest way to explain that link is thru interior automorphisms.

More precisely, given a **non-zero** quaternion $q=\alpha+\beta i+\gamma j+\delta k$, we can consider the interior automorphism:

$$\begin{array}{rccc}\lambda_q: & \mathbf{H} & \to & \mathbf{H} \\ & p & \mapsto & (q\,)p(q^{-1})\end{array}$$

These objects have several fundamental properties: $\lambda_{qq'}=\lambda_q\circ\lambda_{q'}$ and $\lambda_q(q)=q$, λ_q leaves $\mathbf{R}$ invariant (since reals commute with all quaternions), and λ_q respects the norm on $\mathbf{H}$.

It is interesting to see λ_q as an $\mathbf{R}$-linear function on $\mathbf{H}$. As it preserves the norm, it preserves the scalar product, and hence $\lambda_q\in \mathrm{O}(4,\mathbf{H})$. Then, as it leaves $\mathbf{R}$ globally invariant, it must leave its orthogonal (*i.e.* the unreals) globally invariant.

Consider now the matrix of λ_q ; expressed in the canonical basis $\mathcal{C}=(1,i,j,k)$ that matrix is:

$$\mathcal{M}(\lambda_q,\mathcal{C},\mathcal{C})=\frac{1}{\|q\|^2}\begin{bmatrix}\alpha^2+\beta^2+\gamma^2+\delta^2 & 0 & 0 & 0\\ 0 & \alpha^2+\beta^2-\gamma^2-\delta^2 & -2\alpha\delta+2\beta\gamma & +2\alpha\gamma+2\beta\delta\\ 0 & +2\alpha\delta+2\beta\gamma & \alpha^2-\beta^2+\gamma^2-\delta^2 & -2\alpha\beta+2\gamma\delta\\ 0 & -2\alpha\gamma+2\beta\delta & +2\alpha\beta+2\gamma\delta & \alpha^2-\beta^2-\gamma^2+\delta^2\end{bmatrix}$$

It is quite obvious[3] that $\mathcal{Q}:\left[\mathbf{R}^4-\{0\}\to \mathrm{M}(\mathbf{R},4,4);\,q\mapsto\mathcal{M}(\lambda_q,\mathcal{C},\mathcal{C})\right]$ is continuous, and a group homomorphism. As we have seen, $\mathcal{Q}\left(\mathbf{R}^4-\{0\}\right)\subset \mathrm{O}(4,\mathbf{R})$, and as $\mathcal{Q}(1)=\mathrm{I}_4$, the identity matrix[4] on $\mathbf{H}=\mathbf{R}^4$, $\mathcal{Q}\left(\mathbf{R}^4-\{0\}\right)$ must actually be included in the connected component of

[2] We will define this in the next chapter.

[3] We will note $\mathrm{M}(U,n,m)$ the set of matrices, n rows by m columns, with elements in U.

[4] More generally, we will denote by I_n the identity matrix on $\mathbf{R}^n$.

I_4 in $\mathrm{O}(4,\mathbf{R})$, and that is $\mathrm{SO}(4,\mathbf{R})$, *i.e.*, λ_q is a rotation on $\mathbf{R}^4$, and hence on $\mathbf{R}$, where it is the identity I_1, and thus must also be a rotation on $\{0\}\times\mathbf{R}^3$, *i.e.* the unreals. We can find a far simpler (if somewhat tedious) proof of that by simply computing the determinant of $\mathcal{M}\left(\lambda_q,c,c\right)$, which of course turns out to be 1 (also see next section)...

We can therefore extract a rotation matrix on $\mathbf{R}^3$ from $\mathcal{M}\left(\lambda_q,c,c\right)$:

$$\rho_q=\frac{1}{\alpha^2+\beta^2+\gamma^2+\delta^2}\begin{bmatrix}\alpha^2+\beta^2-\gamma^2-\delta^2 & -2\alpha\delta+2\beta\gamma & +2\alpha\gamma+2\beta\delta\\ +2\alpha\delta+2\beta\gamma & \alpha^2-\beta^2+\gamma^2-\delta^2 & -2\alpha\beta+2\gamma\delta\\ -2\alpha\gamma+2\beta\delta & +2\alpha\beta+2\gamma\delta & \alpha^2-\beta^2-\gamma^2+\delta^2\end{bmatrix}$$

Let us introduce $\mathcal{R}:\left[\mathbf{R}^4-\{0\}\to\mathrm{M}(\mathbf{R},3,3);q\mapsto\rho_q\right]$. It is trivial to see that $\mathcal{Q}$ and $\mathcal{R}$ are both C^∞ (because they are rational). It is important to note that they are both $\mathbf{R}$-homogeneous of degree 0, which means that given any non-zero real number x, λ_q and λ_{xq} are identical, and therefore yield identical rotations (*i.e.* $\rho_q=\rho_{xq}$).

A fundamental result is that $\mathcal{R}$ is surjective. There are at least two well-known ways to prove this.

The easiest way also has the advantage of being completely constructive: we just compute the elements of the rotation ρ_q.

This is possible because we always know one invariant vector. Indeed (as an immediate consequence of $\lambda_q(q)=q$):

$$\rho_q\begin{bmatrix}\beta\\ \gamma\\ \delta\end{bmatrix}=\begin{bmatrix}\beta\\ \gamma\\ \delta\end{bmatrix}$$

Furthermore, the angle, $\theta\in[0;\pi]$, is given by considering the trace of ρ_q:

$$1+2\cos(\theta)=\frac{3\alpha^2-\beta^2-\gamma^2-\delta^2}{\alpha^2+\beta^2+\gamma^2+\delta^2}$$

We now exploit the homogeneity of $\mathcal{R}$, which implies that $\mathcal{R}\left(\mathbf{H}-\{0\}\right)=\mathcal{R}\left(\mathbf{S}^3\right)$, and therefore that we can restrict our search to *unit* quaternions. For unit quaternions, the trace relation simplifies to $1+\cos(\theta)=2\alpha^2$.

Therefore, the identity rotation I_3 is associated with $q=\pm1$ (which we already knew), and these unit quaternions only.

Let $\left(\vec{i},\vec{j},\vec{k}\right)$ be the canonical basis of $\mathbf{R}^3$. Consider now a rotation $r\neq\mathrm{I}_3$ (hence $\theta\in]0;\pi]$), it possesses a unique rotation axis, and a unique unit vector $\vec{u}=x\vec{i}+y\vec{j}+z\vec{k}$ directing that axis such that $r(\vec{a})=\left(1-\cos(\theta)\right)(\vec{u}\cdot\vec{a})\vec{u}+\sin(\theta)(\vec{u}\wedge\vec{a})+\cos(\theta)\vec{a}$ for all $\vec{a}\in\mathbf{R}^3$. It follows that r is associated with the two unit quaternions

$$q = \pm \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \frac{x}{\sin\left(\frac{\theta}{2}\right)} \\ \frac{y}{\sin\left(\frac{\theta}{2}\right)} \\ \frac{z}{\sin\left(\frac{\theta}{2}\right)} \end{bmatrix}$$

and these two unit quaternions only.

The second method is non-constructive, but has the advantage of highlighting the regularity of the connection between rotations and quaternions, which is harder to read using the first method.

We once again exploit the homogeneity of $\mathcal{R}$ and use unit quaternions. Given that we know that in fact $\mathcal{R}(\mathbf{S}^3) \subset \mathrm{SO}(3,\mathbf{R})$, we can consider $\mathcal{R}\big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ which is C^∞ (because it is rational). It is slightly tedious, but possible, to prove that in fact $\mathcal{R}\big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ is a local diffeomorphism at 1. It is also a group homomorphism (stemming from the fact that $\lambda_{qq'} = \lambda_q \circ \lambda_{q'}$). Since in a connected topological group, every neighborhood of the neutral element is a generator of the whole group ([G. Pichon (1973), p 31]), $\mathcal{R}\big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ is surjective upon the connected component of $\mathrm{I}_3 = \mathcal{R}(1)$ in $\mathrm{SO}(3,\mathbf{R})$, *i.e.* upon $\mathrm{SO}(3,\mathbf{R})$, and of course is everywhere a local diffeomorphism (though it is of course not a global diffeomorphism).

Combining these two approaches, one finds a global C^∞-diffeomorphism between $\mathrm{SO}(3,\mathbf{R})$ and $\mathbf{RP}^3$ (which is nothing more than $\mathbf{S}^3$ where every couple of opposite points have been identified).

Another thing worth noting is that $\mathcal{R}\big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ is more than just a locally diffeomorphic bijection. If we call $\sigma_{\mathbf{S}^3}$ the positive Borel measure on $\mathbf{S}^3$ induced by $\mathbf{H} = \mathbf{R}^4$ and $\sigma_{\mathrm{SO}(3,\mathbf{R})}$ that induced on $\mathrm{SO}(3,\mathbf{R})$ by $\mathrm{M}(\mathbf{R},3,3)$ (by assimilation of the rotations with their matrix in the canonical basis of $\mathbf{R}^3$), seen as $\mathbf{R}^9$, then we can compute[5] that $\mathcal{R}^*\sigma_{\mathrm{SO}(3,\mathbf{R})} = 16\sqrt{2}\,\sigma_{\mathbf{S}^3}$. Furthermore, $\mathcal{R}\big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ actually has no critical point.

[5] A fact that is supposed to be found, but is not, in [C.W. Misner, K.S. Thone, J.A. Wheeler (1973)].

5- Miscellany

As we have seen, the main power of the quaternions is their ability to pleasantly parameter $\mathrm{SO}(3,\mathbf{R})$. It should be said that what is, perhaps their greatest strengths in this regard, with respect to other parameterization of $\mathrm{SO}(3,\mathbf{R})$ such as Euler angles, is that $\mathcal{R}\Big|_{\mathbf{S}^3}^{\mathrm{SO}(3,\mathbf{R})}$ has no critical points (no "Gimbal Lock"), and that the composition of rotations is extremely simple to compute in terms of the parameter. Also and they can be shown to allow interpolations of orientations under constraints (such has having one axis stay "horizontal").

Quaternions also allow a nice parameterization of $\mathrm{SO}(4,\mathbf{R})$ ([M. Berger (1990)] the application $\mathbf{S}^3\times\mathbf{S}^3\to\mathrm{SO}(4,\mathbf{R}),\ (s,r)\mapsto[q\mapsto sq\bar{r}]$ is a continuous group homomorphism, surjective, with kernel $\{(1,1),(-1,-1)\}$).

Quaternions have other uses, though. For instance, they can be also be used to parameter $\mathrm{SU}(2,\mathbf{C})$. More precisely, an isomorphism exists between $\{0\}\times\mathbf{S}^3$ and $\mathrm{SU}(2,\mathbf{C})$ (consider, the application

$$\begin{array}{rccc}\Psi: & \mathbf{H} & \to & \mathrm{M}(\mathbf{C},2,2)\\ & q=\alpha+\beta i+\gamma j+\delta k & \mapsto & \begin{bmatrix}u & -\bar{v}\\ v & \bar{u}\end{bmatrix}\end{array}$$

with $u=\alpha+\delta i$ and $v=\gamma+\beta i$ is a ring isomorphism from $\mathbf{H}$ on a sub-ring of $\mathrm{M}(\mathbf{C},2,2)$, which induces an isomorphism). There are also applications of quaternions to the Riemann sphere ([J. Lelong-Ferrand, J.M. Arnaudiès (1978)]).

It should be mentioned that research exists to find more efficient algorithm for the product of quaternions ([T. Howell, J.C. Lafon (1975)]), but has so far not reached a conclusion, one way or the other.

Given the power of the quaternions, the question naturally arises as to whether something similar can be done for rotations on spaces of higher dimensions (the multiplication being commutative on the reals and complex numbers, interiors automorphisms are just the identity). The answer to that question is partly positive, but it should be now stated that the right tool, in general, for that problem turns out to be Clifford algebras rather than Cayley algebras.

When we turn to the octonions, the multiplication is not only not associative, it is no longer even associative. Fortunately, the sub-algebra engendered by any two elements (and the unity) is still associative, and therefore interior automorphism do not depend on the order in which the products are carried out. The interesting fact is that, as with the quaternions, the interior automorphisms leave $\mathbf{R}$ invariant, and induce a rotation, on $\mathbf{R}^7$ this time. The catch is that $\mathrm{SO}(7,\mathbf{R})$ is a 21-dimentional manifold, whereas the interior automorphisms we just described only have 7 degrees of freedom. In short, we do not get all the rotations on $\mathbf{R}^7$ by this method. It is still useful, though, for theoretical purposes.

Beyond the even the octonions, the hexadecimalions have two different flavors of interior automorphism, $p \mapsto ((q)p)(q^{-1})$ and $p \mapsto (q)(p(q^{-1}))$, neither of which is, in general, a rotation (on either $\mathbf{R}^{16}$ or $\mathbf{R}^{15}$). The average of the two isn't a rotation either, by the way...

Interior automorphisms having apparently reached the limits of their usefulness, we turn now to something else, with the same objects. It turns out that we can find rotations with even simpler constructions!

Let $x = \alpha \in \mathbf{R}$, then $\mathrm{M}_x = \mathcal{M}([y \mapsto xy],1,1) = \mathcal{M}([y \mapsto yx],1,1) = [\alpha]$, hence ${}^{\mathrm{t}}\mathrm{M}_x \mathrm{M}_x = |x| \mathrm{I}_1$, and $\det(\mathrm{M}_x) = x$. Therefore if $|x| = 1$, we find that $\mathrm{M}_x \in \mathrm{O}(1,\mathbf{R})$, and we of course get all two elements of $\mathrm{O}(1,\mathbf{R})$ that way... but $\mathrm{M}_x \in \mathrm{SO}(1,\mathbf{R})$ only if $x = 1$! Obviously, given $x \in \mathbf{R}$ and $x' \in \mathbf{R}$, $\mathrm{M}_{xx'} = \mathrm{M}_x \mathrm{M}_{x'} = \mathrm{M}_{x'} \mathrm{M}_x$.

Let $c = \alpha + \beta i \in \mathbf{C}$, then $\mathrm{M}_c = \mathcal{M}([z \mapsto cz],(1,i),(1,i)) = \mathcal{M}([z \mapsto zc],(1,i),(1,i)) = \begin{bmatrix} \alpha & -\beta \\ +\beta & \alpha \end{bmatrix}$, hence ${}^{\mathrm{t}}\mathrm{M}_c \mathrm{M}_c = |c| \mathrm{I}_2$, and $\det(\mathrm{M}_c) = \alpha^2 + \beta^2$. Therefore if $|c| = 1$, $\mathrm{M}_c \in \mathrm{SO}(2,\mathbf{R})$, and we get all rotations on $\mathbf{R}^2$ that way, as is well-known. And given $c \in \mathbf{C}$ and $c' \in \mathbf{C}$ we still have $\mathrm{M}_{cc'} = \mathrm{M}_c \mathrm{M}_{c'} = \mathrm{M}_{c'} \mathrm{M}_c$.

Let now $q = \alpha + \beta i + \gamma j + \delta k \in \mathbf{H}$, then:

$$\mathrm{M}_q^G = \mathcal{M}([p \mapsto qp],(1,i,j,k),(1,i,j,k)) = \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ +\beta & \alpha & -\delta & +\gamma \\ +\gamma & +\delta & \alpha & -\beta \\ +\delta & -\gamma & +\beta & \alpha \end{bmatrix}$$

and

$$\mathrm{M}_q^D = \mathcal{M}([p \mapsto pq],(1,i,j,k),(1,i,j,k)) = \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ +\beta & \alpha & +\delta & -\gamma \\ +\gamma & -\delta & \alpha & +\beta \\ +\delta & +\gamma & -\beta & \alpha \end{bmatrix}$$

, hence ${}^{\mathrm{t}}\mathrm{M}_q^G \mathrm{M}_q^G = {}^{\mathrm{t}}\mathrm{M}_q^D \mathrm{M}_q^D = |q| \mathrm{I}_4$, and $\det(\mathrm{M}_q^G) = \det(\mathrm{M}_q^D) = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$. Therefore if $|q| = 1$, $\mathrm{M}_q^G \in \mathrm{SO}(4,\mathbf{R})$ and $\mathrm{M}_q^D \in \mathrm{SO}(4,\mathbf{R})$, but we only get a tiny fraction of $\mathrm{SO}(4,\mathbf{R})$ that way.

This, of course can be used as an alternate proof that the interior automorphisms on the quaternions actually induce rotations on $\mathbf{R}^4$.

It is interesting to note that given $q \in \mathbf{H}$ and $q' \in \mathbf{H}$, we still have $\mathrm{M}_{qq'}^G = \mathrm{M}_q^G \mathrm{M}_{q'}^G$ and $\mathrm{M}_{qq'}^D = \mathrm{M}_{q'}^D \mathrm{M}_q^D$, though we now sometimes have $\mathrm{M}_q^G \mathrm{M}_{q'}^G \neq \mathrm{M}_{q'}^G \mathrm{M}_q^G$ and $\mathrm{M}_{q'}^D \mathrm{M}_q^D \neq \mathrm{M}_q^D \mathrm{M}_{q'}^D$.

Turning to the octonions, let $o=\alpha+\beta i+\gamma j+\delta k+\varepsilon e'+\zeta i'+\eta j'+\theta k'\in\mathbf{O}$, then:

$$\mathrm{M}_o^G=\mathcal{M}\left(\left[o'\mapsto oo'\right],(1,i,j,k,e',i',j',k'),(1,i,j,k,e',i',j',k')\right)=\begin{bmatrix}\alpha&-\beta&-\gamma&-\delta&-\varepsilon&-\zeta&-\eta&-\theta\\+\beta&\alpha&-\delta&+\gamma&-\zeta&+\varepsilon&+\theta&-\eta\\+\gamma&+\delta&\alpha&-\beta&-\eta&-\theta&+\varepsilon&+\zeta\\+\delta&-\gamma&+\beta&\alpha&-\theta&+\eta&-\zeta&+\varepsilon\\+\varepsilon&+\zeta&+\eta&+\theta&\alpha&-\beta&-\gamma&-\delta\\+\zeta&-\varepsilon&+\theta&-\eta&+\beta&\alpha&+\delta&-\gamma\\+\eta&-\theta&-\varepsilon&+\zeta&+\gamma&-\delta&\alpha&+\beta\\+\theta&+\eta&-\zeta&-\varepsilon&+\delta&+\gamma&-\beta&\alpha\end{bmatrix}$$

and

$$\mathrm{M}_o^D=\mathcal{M}\left(\left[o'\mapsto o'o\right],(1,i,j,k,e',i',j',k'),(1,i,j,k,e',i',j',k')\right)=\begin{bmatrix}\alpha&-\beta&-\gamma&-\delta&-\varepsilon&-\zeta&-\eta&-\theta\\+\beta&\alpha&+\delta&-\gamma&+\zeta&-\varepsilon&-\theta&+\eta\\+\gamma&-\delta&\alpha&+\beta&+\eta&+\theta&-\varepsilon&-\zeta\\+\delta&+\gamma&-\beta&\alpha&+\theta&-\eta&+\zeta&-\varepsilon\\+\varepsilon&-\zeta&-\eta&-\theta&\alpha&+\beta&+\gamma&+\delta\\+\zeta&+\varepsilon&-\theta&+\eta&-\beta&\alpha&-\delta&+\gamma\\+\eta&+\theta&+\varepsilon&-\zeta&-\gamma&+\delta&\alpha&-\beta\\+\theta&-\eta&+\zeta&+\varepsilon&-\delta&-\gamma&+\beta&\alpha\end{bmatrix}$$

, hence ${}^{\mathrm{t}}\mathrm{M}_o^G\mathrm{M}_o^G={}^{\mathrm{t}}\mathrm{M}_o^D\mathrm{M}_o^D=|o|\mathrm{I}_8$, and $\det\left(\mathrm{M}_o^G\right)=\det\left(\mathrm{M}_o^D\right)=\left(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2+\zeta^2+\eta^2+\theta^2\right)^4$. Therefore, if $|o|=1$, $\mathrm{M}_o^G\in\mathrm{SO}(8,\mathbf{R})$ and $\mathrm{M}_o^D\in\mathrm{SO}(8,\mathbf{R})$. Again, we only get a very tiny fraction of $\mathrm{SO}(8,\mathbf{R})$ that way.

Also, and contrary to the case for the real numbers, the complex numbers and the quaternions, in general $\mathrm{M}_{oo'}^G\neq\mathrm{M}_o^G\mathrm{M}_{o'}^G$ and $\mathrm{M}_{oo'}^D\neq\mathrm{M}_{o'}^D\mathrm{M}_o^D$, due to the non-associativity of the product on $\mathbf{O}$. For instance, $i'e'=-i$, but $\mathrm{M}_{i'}^G\mathrm{M}_{e'}^G\neq\mathrm{M}_{-i}^G$.

If we try to do the same thing with hexadecimalions, we find that neither $\mathcal{M}\left(\left[l\mapsto hl\right],(1,i,j,k,e',i',j',k'',e'',i'',j'',k'',e''',i''',j''',k'''),(1,i,j,k,e',i',j',k'',e'',i'',j'',k'',e''',i''',j''',k''')\right)$ nor its right-hand version are rotation in general, even if $\|l\|=1$. That trail ends here as well!

Chapter 2 Building the Quaternions

1- What to find here

This chapter, except for Section 5, only consists of well-known classical results ([N. Bourbaki (A)], [S. Lang (1991)],...). Some have been slightly restated (usually with simplifications) from their original sources, but hardly anything new is presented here. In case the sources disagree on definitions, [N. Bourbaki (A)] will take precedence.

2- Cayley algebra, alternative algebra

Some of the structures we will be considering will not even be associative. To save what may be, a weaker structure, which is interesting in its own right is presented first. An algebra $\mathbf{E}$ is said to be alternative if the following trilinear application, known as the associator of $\mathbf{E}$, is alternating (which means its value is zero if two of its arguments are identical):

$$\begin{array}{rccc} \mathrm{a}: & \mathbf{E}\times\mathbf{E}\times\mathbf{E} & \to & \mathbf{E} \\ & (x,y,z) & \mapsto & x*(y*z)-(x*y)*z \end{array}$$

This notion is interesting as, though an alternative algebra is not as wieldy as an associative algebra, it is such that every sub-algebra engendered by any two elements *is* associative. It also implies that an alternative algebra is a division algebra (which means that for any $x\in\mathbf{E}$, $x\neq 0$, the applications $\mathbf{E}\to\mathbf{E}; y\mapsto x*y$ and $\mathbf{E}\to\mathbf{E}; y\mapsto y*x$ are bijective, or that elements are "simplifiable"). In particular the inverse of a non-zero element (if it exists) is unique in such an algebra.

The meat of this chapter is the following structure.

Let $\mathbf{A}$ be a commutative ring, and $\mathbf{E}$ an algebra over $\mathbf{A}$, not necessarily commutative or associative, but having a unit element e (remember that since $\mathbf{E}$ is an $\mathbf{A}$-algebra, then $(\forall\lambda\in\mathbf{A})(\forall x\in\mathbf{E})\ \lambda\cdot x=(\lambda\cdot e)*x=x*(\lambda\cdot e)$).

A conjugation over $\mathbf{E}$ is any (there may be none) bijective, $\mathbf{A}$-linear, function $\sigma:\mathbf{E}\to\mathbf{E}$ such that:

1) $\sigma(e)=e$.
2) $(\forall(x,y)\in\mathbf{E}^2)\ \ \sigma(x*y)=\sigma(y)*\sigma(x)$ (*beware* the inversion of x and y!).
3) $(\forall x\in\mathbf{E})\ \ (x+\sigma(x))\in\mathbf{A}\cdot e$ and $(\forall x\in\mathbf{E})\ \ (x*\sigma(x))\in\mathbf{A}\cdot e$.

These properties imply[6] $(\forall x\in\mathbf{E})\ \ x*\sigma(x)=\sigma(x)*x$, and[7] $(\forall x\in\mathbf{E})\ \ \sigma\circ\sigma(x)=x$.

We will also write $\bar{x}$ for $\sigma(x)$.

[6] $(x+\sigma(x))\in\mathbf{A}\cdot e\Rightarrow x*\sigma(x)=x*(x+\sigma(x))-x*x=(x+\sigma(x))*x-x*x=\sigma(x)*x$.

[7] Given $x\in\mathbf{E}$, there exists $\alpha\in\mathbf{A}$ such that $x+\sigma(x)=\alpha\cdot e$; the $\mathbf{A}$-linearity of σ then implies $\sigma(x)+\sigma\circ\sigma(x)=\sigma(x+\sigma(x))=\alpha\cdot\sigma(e)$, and finally, $\sigma(e)=e$.

If $\mathbf{E}$ is such an algebra, and if σ is a conjugation over $\mathbf{E}$, the structure $(\mathbf{E},+,*,\cdot,\sigma)$ is said to be a cayley algebra over $\mathbf{A}$. On such a structure, it is convenient to consider the cayley trace and cayley norm (an unfortunate misnomer as it is actually quadratic...), defined respectively by $\mathcal{T}_{\mathbf{E}}(x)=x+\sigma(x)$ and $\mathcal{N}_{\mathbf{E}}(x)=x*\sigma(x)$.

Note that if $(\mathbf{E},+,*)$ has no zero divisors, for instance if it is a field, then $\mathcal{N}_{\mathbf{E}}(x)=0$ if and only if $x=0$.

We have the important relations:

$$\mathcal{T}_{\mathbf{E}}(\sigma(x))=\mathcal{T}_{\mathbf{E}}(x)$$
$$\mathcal{N}_{\mathbf{E}}(\sigma(x))=\mathcal{N}_{\mathbf{E}}(x)$$
$$\mathcal{T}(x*y)=\mathcal{T}(y*x)$$
$$\mathcal{T}_{\mathbf{E}}(x*\sigma(y))=\mathcal{T}_{\mathbf{E}}(y*\sigma(x))=\mathcal{T}_{\mathbf{E}}(x)*\mathcal{T}_{\mathbf{E}}(y)-\mathcal{T}_{\mathbf{E}}(x*y)=\mathcal{N}_{\mathbf{E}}(x+y)-\mathcal{N}_{\mathbf{E}}(x)-\mathcal{N}_{\mathbf{E}}(y)$$

It is interesting to note that $\mathcal{T}(x*y)=\mathcal{T}(y*x)$ regardless of whether or not $\mathbf{E}$ is associative or commutative. For the cayley norm, no such broad result seem to hold[8]; however if $\mathbf{E}$ is alternative, then we also have $\mathcal{N}_{\mathbf{E}}(x*y)=\mathcal{N}_{\mathbf{E}}(x)\mathcal{N}_{\mathbf{E}}(y)$.

Finally, the following lemma will be useful for our purposes:

Lemma (Complexoïd): # Given $x\in\mathbf{E}$, $\mathrm{Vect}_{\mathbf{A}}(e,x)$, the $\mathbf{A}$-module spanned by x and e, is stable for $*$; it is a sub-cayley algebra of $\mathbf{E}$ which is both associative and commutative. If $x\notin\mathbf{A}\cdot e$, let $y=\alpha\cdot e+\beta\cdot x$, $\mathrm{M}_y^G=\mathcal{M}\left([u\mapsto y*u],(e,x),(e,x)\right)$ and $\mathrm{M}_y^D=\mathcal{M}\left([u\mapsto u*y],(e,x),(e,x)\right)$; then (with $\mathrm{T}\cdot e=\mathcal{T}_{\mathbf{E}}(x)$ and $\mathrm{N}\cdot e=\mathcal{N}_{\mathbf{E}}(x)$)

$$\mathrm{M}_y=\mathrm{M}_y^G=\mathrm{M}_y^D=\begin{bmatrix}\alpha & -\beta\mathrm{N}\\ \beta & \beta\mathrm{T}+\alpha\end{bmatrix}$$

. Given $z\in\mathrm{Vect}(e,x)$, we have $\mathrm{M}_{y*z}=\mathrm{M}_y\,\mathrm{M}_z=\mathrm{M}_z\,\mathrm{M}_y=\mathrm{M}_{z*y}$. $

This is a simple consequence of the fact that $x*x=\mathrm{T}\cdot x-\mathrm{N}\cdot e$, with $\mathrm{T}\cdot e=\mathcal{T}_{\mathbf{E}}(x)$ and $\mathrm{N}\cdot e=\mathcal{N}_{\mathbf{E}}(x)$! $

This lemma allows us, in particular, to define unambiguously the n-th power, with $n\in\mathbf{N}$, of any $x\in\mathbf{E}$ by the usual recursion rules, we will write the result, as usual x^n. It also trivially induces the following scholie:

Scholie (Powers): # Given $x\in\mathbf{E}$, and $n\in\mathbf{N}$, $x^n\in\mathrm{Vect}_{\mathbf{A}}(e,x)$ and $\mathcal{N}_{\mathbf{E}}\left(x^n\right)=\left(\mathcal{N}_{\mathbf{E}}(x)\right)^n$. $

3- The Cayley doubling procedure

It should be noted that this is simply the plain vanilla version of the doubling process[9]; it will suffice here, however.

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[8] Indeed, we have seen that such an equality does not hold for hexadecimalions!

[9] The general procedure involves abitrary coefficients which parameterize the operations.

Let $\mathbf{A}$ be a commutative ring, and $(\mathbf{E},+,*,\cdot,\sigma)$ a cayley algebra over $\mathbf{A}$, not necessarily commutative or associative, with unit element e. Let $\mathbf{F}=\mathbf{E}\times\mathbf{E}$ and $e_{\mathbf{F}}=(e,0)\in\mathbf{F}$; furthermore, let:

$$\begin{array}{rccc} +_{\mathbf{F}}: & \mathbf{F}\times\mathbf{F} & \to & \mathbf{F} \\ & ((x,y),(x',y')) & \mapsto & (x+x',y+y') \end{array}$$

$$\begin{array}{rccc} *_{\mathbf{F}}: & \mathbf{F}\times\mathbf{F} & \to & \mathbf{F} \\ & ((x,y),(x',y')) & \mapsto & (x*x'-\overline{y'}*y,y*\overline{x'}+y'*x) \end{array}$$

$$\begin{array}{rccc} \cdot_{\mathbf{F}}: & \mathbf{A}\times\mathbf{F} & \to & \mathbf{F} \\ & (\lambda,(x,y)) & \mapsto & (\lambda\cdot x,\lambda\cdot y) \end{array}$$

$$\begin{array}{rccc} \sigma_{\mathbf{F}}: & \mathbf{F} & \to & \mathbf{F} \\ & (x,y) & \mapsto & (\sigma(x),-y) \end{array}$$

Proposition (Structure): # $(\mathbf{F},+_{\mathbf{F}},*_{\mathbf{F}},\cdot_{\mathbf{F}})$ is an $\mathbf{A}$-algebra, with unit $e_{\mathbf{F}}$, and $\sigma_{\mathbf{F}}$ is a conjugation over $\mathbf{F}$; $\mathbf{F}$ is associative if and only if $\mathbf{E}$ is both associative and commutative; $\mathbf{F}$ is alternative if and only $\mathbf{E}$ is associative. Furthermore, $\mathcal{T}_{\mathbf{F}}((x,y))=\mathcal{T}_{\mathbf{E}}(x)$ and $\mathcal{N}_{\mathbf{F}}((x,y))=\mathcal{N}_{\mathbf{E}}(x)+\mathcal{N}_{\mathbf{E}}(y)$. $

Keep in mind that since $\mathbf{F}$ is also an $\mathbf{A}$-algebra then $(\forall\lambda\in\mathbf{A})(\forall(x,y)\in\mathbf{F})\ \lambda\cdot_{\mathbf{F}}(x,y)=(\lambda\cdot_{\mathbf{F}}e_{\mathbf{F}})*_{\mathbf{F}}(x,y)=(x,y)*_{\mathbf{F}}(\lambda\cdot_{\mathbf{F}}e_{\mathbf{F}})$. It is interesting to note that, if $\mathbf{E}$ is associative, we still have $\mathcal{N}_{\mathbf{F}}((x,y)*_{\mathbf{F}}(x',y'))=\mathcal{N}_{\mathbf{F}}((x,y))\,\mathcal{N}_{\mathbf{F}}((x',y'))$, even if $\mathbf{F}$ is not associative.

Given the proposition, we can (and will) identify $\mathbf{E}$ with $\mathbf{E}\times\{0_{\mathbf{E}}\}$. Alternatively, we can identify $\mathbf{F}$ with a superset of $\mathbf{E}$. It is also possible to identify $\mathbf{A}$ with a subset of $\mathbf{E}$ (and hence of $\mathbf{F}$ as well), in that case we have noted that all elements of $\mathbf{A}$ commute with all elements of $\mathbf{E}$, for the multiplication in $\mathbf{E}$, as well as with all elements of $\mathbf{F}$, for the multiplication in $\mathbf{F}$, even though $\mathbf{E}$ or $\mathbf{F}$ might not be commutative. With this identification, $\mathcal{T}$ and $\mathcal{N}$ have value in $\mathbf{A}$.

4- R, C, H, O, X…

We now consider $\mathbf{A}=\mathbf{R}$ and $\mathbf{E}=\mathbf{R}$, with $\sigma(x)=x$ and $e=1$, then $\mathcal{N}_{\mathbf{R}}(x)=x^2$ is always positive (and zero if and only $x=0$, as $\mathbf{R}$ is a field). When we build $\mathbf{F}$ as above, we get exactly $\mathbf{C}$, and $\sigma_{\mathbf{F}}$ is the usual conjugation on $\mathbf{C}$. We define $i=(0;1)$, and as stated earlier, we identify $\mathbf{R}$ with $\mathbf{R}\times\{0\}$. As is well known, $\mathbf{C}$ is a commutative field, in particular, real numbers commute with complex numbers. Due to our identifications, $\mathcal{T}_{\mathbf{C}}$ and $\mathcal{N}_{\mathbf{C}}$ have values in $\mathbf{R}$, and actually, if $z=x+iy$ then $\mathcal{N}_{\mathbf{C}}(z)=\mathcal{N}_{\mathbf{R}}(x)+\mathcal{N}_{\mathbf{R}}(y)=x^2+y^2=|z|^2\geq 0$, and $\mathcal{N}_{\mathbf{C}}(z)=0$ if and only if $z=0$. We lose some of the original properties of $\mathbf{R}$ as we build $\mathbf{C}$, for instance we lose the existence of an order compatible with the multiplication; we do get new and interesting properties at the same time, of course.

Let's do the doubling again, this time with $\mathbf{A}=\mathbf{R}$ and $\mathbf{E}=\mathbf{C}$, with the usual conjugation, and this time we get exactly $\mathbf{H}$, the conjugation being the same as defined earlier, given the definition of $j=(0;1)$ and $k=(0;i)$, and the identification of $\mathbf{C}$ with $\mathbf{C}\times\{0\}$. Once again, we note that, as predicted, for quaternion multiplication, real numbers commute with quaternions, though some quaternions do not commute (for instance $i*j\neq j*i$). As already stated $\mathbf{H}$ is a (non-commutative) field. Once again, due to our new identifications, $\mathcal{T}_{\mathbf{H}}$ and $\mathcal{N}_{\mathbf{H}}$ have values in $\mathbf{R}$, and actually, $\mathcal{N}_{\mathbf{H}}$ is always positive and $\mathcal{N}_{\mathbf{H}}(q)=0$ if and only if $q=0$. We keep loosing original properties, most notably the commutativity, when we go from $\mathbf{C}$ to $\mathbf{H}$, but the new properties we gain, notably the link with rotations in $\mathbf{R}^3$, which we saw earlier, still makes it worthwhile. We also see that $\mathcal{T}_{\mathbf{H}}(q)=2\,\mathrm{Re}(q)$ and $\mathcal{N}_{\mathbf{H}}(q)=\|q\|^2=|q|^2$, as defined earlier.

There being not such thing as too much of a good thing, let's do the doubling once again, this time with $\mathbf{A}=\mathbf{R}$ and $\mathbf{E}=\mathbf{H}$, and the conjugation just built on $\mathbf{H}$. What the process yields this time is known as the set of (Cayley) octonions, whose symbol is $\mathbf{O}$. We, as is now usual, identify $\mathbf{H}$ with $\mathbf{H}\times\{0\}$. Yet again, we note that, for octonion multiplication, real numbers commute with octonions, though some octonions do not commute (as some quaternions already do not commute). Yet again, due to our new identifications, $\mathcal{T}_{\mathbf{O}}$ and $\mathcal{N}_{\mathbf{O}}$ have values in $\mathbf{R}$, and actually, $\mathcal{N}_{\mathbf{O}}$ is always positive and $\mathcal{N}_{\mathbf{O}}(o)=0$ if and only if $o=0$. The situation keeps deteriorating, though, as this time the algebra is not associative anymore (but it is still associative). Octonions do have uses, apart from being an example of a non-associative algebra. They can be used to find a basis of non-vanishing vector field on $\mathbf{S}^7$ (the euclidian unit sphere in $\mathbf{R}^8$), in the same way quaternions can be used to find one on $\mathbf{S}^3$, and complexes are used to find one on $\mathbf{S}^1$. They also see use in theoretical physics ([G. Dixon (1994)]). Octonions still are a division algebra, and non-zero octonions O have $[\mathcal{N}_{\mathbf{O}}(O)]^{-1}\sigma_{\mathbf{O}}(O)$ for inverse. Despite the non-associativity of the multiplication, we still have $\mathcal{N}_{\mathbf{O}}(o*o')=\mathcal{N}_{\mathbf{O}}(o)\,\mathcal{N}_{\mathbf{O}}(o')$, since the multiplication *is* associative on $\mathbf{H}$.

We can keep doubling *ad nauseam,* but things really get unwieldy. At the stage after octonions, the hexadecimalions, $\mathbf{X}$, the algebra is not even alternating. This author does not know of any use the ulterior echelons may have been put to, if any.

5- The full Cayley ladder all at once

One might wonder if the whole doubling procedure might be "carried out to infinity". As it turns out, it can, after a fashion. We will present here a special version[10] of the global object, for simplicity.

Let $\mathbf{A}$ be a commutative ring, whose unit element will be called e.

Let us call $\mathcal{A}_0=\mathbf{A}$ and σ_0 the identity over $\mathbf{A}$. It is quite obvious that $(\mathbf{A},+,\times,\times,\sigma)$ is a cayley algebra over $\mathbf{A}$. Using the doubling procedure, we build $\mathcal{A}_1=\mathbf{A}\times\mathbf{A}$ and σ_1, and by induction we build $\mathcal{A}_n$ and σ_n for all $n\in\mathbf{N}$.

[10] That is, the object built by our "plain vanilla" version of the Cayley doubling procedure.

Consider $\mathbf{A}[X]$ the set of polynomials (in one indeterminate X) with coefficients in $\mathbf{A}$. We already have an $\mathbf{A}$-algebra structure, which we will denote by $(\mathbf{A}[X],+,\cdot,\times)$ and is the usual commutative algebra. We readily identify $\mathcal{A}_0=\mathbf{A}$ with constant polynomials, thru an homomorphism of $\mathbf{A}$-modules $\mathcal{J}_0$. It is trivial to see that $\mathcal{A}_n$ identifies with polynomials of degree less or equal to 2^n-1, thru the trivial $\mathbf{A}$-modules isomorphism $\mathcal{J}_n$. Let us call $I_n:\mathcal{A}_n\to\mathcal{A}_{n+1},x\mapsto(x,0)$ the canonical identification. Then $(\forall n\in\mathbf{N})\,\mathcal{J}_{n+1}\circ I_n=\mathcal{J}_n$, which means our identifications are all coherent.

So every element of every rung of the Cayley "ladder", build by successively doubling the preceding rung and begun by $\mathbf{A}$, *a finite number of times*, can be identified uniquely with some polynomial with coefficients in $\mathbf{A}$, and conversely every element of $\mathbf{A}[X]$ can be seen a some unique element of the Cayley ladder. As the multiplication we will define differs, in general, from the polynomial multiplication, we will choose a new symbol for our construction.

Let $\mathbb{C}(\mathbf{A})$ be some set equipotent to $\mathbf{A}[X]$, the set of polynomials in one indeterminate X over $\mathbf{A}$, thru a bijection $\mathfrak{I}:\mathbb{C}(\mathbf{A})\mapsto\mathbf{A}[X]$. This bijection induces an $\mathbf{A}$-module on $\mathbb{C}(\mathbf{A})$, from $(\mathbf{A}[X],+,\cdot)$, which we will denote by $(\mathbb{C}(\mathbf{A}),+,\cdot)$, and we identify $\mathcal{A}_n$ with $\mathfrak{I}^{-1}\left(\mathcal{J}_n\left(\mathcal{A}_n\right)\right)$.

We will now define a multiplication on $\mathbb{C}(\mathbf{A})$, which we will denote by "$*$". Let $\mathfrak{p}\in\mathbb{C}(\mathbf{A})$ and $\mathfrak{q}\in\mathbb{C}(\mathbf{A})$; let $P=\mathfrak{I}(\mathfrak{p})$ and $Q=\mathfrak{I}(\mathfrak{q})$, then there exists (at least) one $n\in\mathbf{N}$ such that $P\in\mathcal{J}_n\left(\mathcal{A}_n\right)$ and $Q\in\mathcal{J}_n\left(\mathcal{A}_n\right)$. We chose the smallest such n. We now find the only $p_n\in\mathcal{A}_n$ such that $P=\mathcal{J}_n\left(p_n\right)$ and the only $q_n\in\mathcal{A}_n$ such that $Q=\mathcal{J}_n\left(q_n\right)$. Finally $\mathfrak{I}(\mathfrak{p}*\mathfrak{q})=\mathcal{J}_n\left(p_n*_{\mathcal{A}_n}q_n\right)$. We note that for all $n'>n$, we do have $P\in\mathcal{J}_{n'}\left(\mathcal{A}_{n'}\right)$ and $Q\in\mathcal{J}_{n'}\left(\mathcal{A}_{n'}\right)$ and there are unique $p_{n'}\in\mathcal{A}_{n'}$ such that $P=\mathcal{J}_{n'}\left(p_{n'}\right)$ and $q_{n'}\in\mathcal{A}_{n'}$ such that $Q=\mathcal{J}_{n'}\left(q_{n'}\right)$, but thanks to the coherence of the identifications we also have $\mathcal{J}_n\left(p_n*_{\mathcal{A}_n}q_n\right)=\mathcal{J}_{n'}\left(p_{n'}*_{\mathcal{A}_{n'}}q_{n'}\right)$.

It is easy to verify that $(\mathbb{C}(\mathbf{A}),+,\cdot,*)$ is an $\mathbf{A}$-algebra. However, in general $\mathfrak{I}(\mathfrak{p}*\mathfrak{q})\neq\mathfrak{I}(\mathfrak{p})\times\mathfrak{I}(\mathfrak{q})$. For instance if $\mathbf{A}=\mathbf{R}$ then $X^1=\mathfrak{I}(i)$, $X^2=\mathfrak{I}(j)$ and $X^3=\mathfrak{I}(k)$, and thus $\mathfrak{I}(i)=X^1\neq X^5=X^2\times X^3=\mathfrak{I}(j)\times\mathfrak{I}(k)$. So $\mathfrak{I}$ is *not*, in general, an algebra isomorphism between $(\mathbb{C}(\mathbf{A}),+,\cdot,*)$ and $(\mathbf{A}[X],+,\cdot,\times)$, as stated earlier.

We likewise define the conjugation σ, and the cayley trace and "norm", over $\mathbb{C}(\mathbf{A})$ thru the identifications $\mathcal{J}_n$, with values in $\mathcal{A}_0$. It is now easy to check that $(\mathbb{C}(\mathbf{A}),+,\cdot,*,\sigma)$ is a cayley algebra over $\mathbf{A}$ (usually not commutative or associative), which, thru the identifications, contains all the rungs of the cayley doubling procedure starting with $\mathbf{A}$. Elements of $\mathbf{A}$ commute with all elements of $\mathbb{C}(\mathbf{A})$, for $*$.

We will shortly use the fact that if $\mathfrak{I}(\mathfrak{p})=\alpha_0+\alpha_1X+\cdots+\alpha_{2^n-1}X^{2^n-1}=\mathcal{J}_n\left(p_n\right)$, then $\mathfrak{I}(\mathfrak{p}*\mathfrak{p})=\mathcal{J}_n\left(p_n*_{\mathcal{A}_n}p_n\right)=\left(\alpha_0^2-\left(\alpha_1^2+\cdots+\alpha_{2^n-1}^2\right)\right)+2\alpha_1\alpha_0X+\cdots+2\alpha_{2^n-1}\alpha_0X^{2^n-1}$; this is simply proved by recurrence.

As a first example of Cayley ladders, let us consider $(\mathbb{C}(\mathbb{Z}/2\mathbb{Z}),+,\times,*,\mathrm{Id})$. It is a commutative and associative cayley algebra, the conjugation being the identity on $\mathbb{C}(\mathbb{Z}/2\mathbb{Z})$; however it has zero divisors, as for instance $\mathfrak{I}^{-1}(1+X)*\mathfrak{I}^{-1}(1+X)=0$, but if $\mathfrak{p}\in\mathbb{C}(\mathbb{Z}/2\mathbb{Z})$ and $\mathfrak{I}(\mathfrak{p})$has an odd number of 1 then $\mathfrak{I}(\mathfrak{p}*\mathfrak{p})=1$.

The second, perhaps more interesting example, is $\mathbb{C}(\mathbf{R})$, which we have actually used already. In that case we can see that if $\mathfrak{a}\in\mathbb{C}(\mathbf{R})$, then $\mathcal{N}_{\mathbb{C}(\mathbf{R})}(\mathfrak{a})$ is always positive, and $\mathcal{N}_{\mathbb{C}(\mathbf{R})}(\mathfrak{a})=0$ if and only if $\mathfrak{a}=0$; furthermore, $\mathfrak{a}\neq 0\Rightarrow\left[\left(\mathcal{N}_{\mathbb{C}(\mathbf{R})}(\mathfrak{a})\right)^{-1}\sigma(\mathfrak{a})\right]*\mathfrak{a}=\mathfrak{a}*\left[\left(\mathcal{N}_{\mathbb{C}(\mathbf{R})}(\mathfrak{a})\right)^{-1}\sigma(\mathfrak{a})\right]=1$. It is also possible in this case to compute square roots! Indeed, let $x\in\mathcal{A}_n$, with $\mathcal{I}_n(x)=A_0+A_1X+\cdots+A_{2^n-1}X^{2^n-1}$; we seek $y\in\mathcal{A}_n$ with $\mathcal{I}_n(y)=\alpha_0+\alpha_1X+\cdots+\alpha_{2^n-1}X^{2^n-1}$ such that $y*y=x$. This amounts to solving, in $\mathbf{R}^{2^n}$ the system:

$$\begin{cases}\alpha_0^2-\left(\alpha_1^2+\cdots\alpha_{2^n-1}^2\right)=A_0\\ 2\alpha_1\alpha_0=A_1\\ \vdots\\ 2\alpha_{2^n-1}\alpha_0=A_{2^n-1}\end{cases}$$

This system is easily solved by considering first the case $x=0$, for which there is a unique solution $y=0$, second the subcase $x\in\mathbf{R}_+^*$ for which there are exactly two solutions given by $y=\pm\sqrt{x}$, third the subcase $x\notin\mathbf{R}$ (if $n\geq 1$, of course) for which there are also exactly two solution given by $\mathcal{I}_n(y)=\alpha_0+\frac{A_1}{2\alpha_0}X+\cdots+\frac{A_{2^n-1}}{2\alpha_0}X^{2^n-1}$ with $\alpha_0=\pm\frac{\sqrt{2}}{2}\sqrt{\mathrm{Re}(x)+|x|}$, and finally the case $x\in\mathbf{R}_-^*$, for which the solutions are all the $y\in\mathcal{A}_n$ such that $\mathcal{R}(y)=0$ and $|y|=|x|$.

This means that the solutions to $y^2=x$, where $x\notin\mathbf{R}_-$ are the same in every rung of the real Cayley ladder (that is, there are exactly two, opposite solutions, belonging to the same rung), and the solution to $y^2=0$ is always $y=0$, in whatever rung of the Cayley ladder. However, solutions to $y^2=x$ for $x\in\mathbf{R}_-^*$ differ depending upon the precise rung: in $\mathbf{R}$ there is no solution, in $\mathbf{C}$ there are exactly two, opposite, solutions, in $\mathbf{H}$ and above there is an innumerable number of solutions (full spheres)!

Note that in any case a y such that $y^2=x$ commutes with x, but that two such solutions need not commute with each other!

At least two topologies are interesting to consider on $\mathbb{C}(\mathbf{R})$: the norm topology induced by the square root of$\mathcal{N}$, the Cayley "norm" on $\mathbb{C}(\mathbf{R})$ (we will write $\|c\|=\sqrt{\mathcal{N}(c)}$), which we will call $\mathcal{T}$, and the strict inductive limit topology ([V.-K. Khoan (1972)]) defined by the rungs $\mathbb{C}_n=\mathcal{I}_n(\mathcal{A}_n)$ of $\mathbb{C}(\mathbf{R})$ on which we consider the norms $q_n=2^{\sup(0,n-2)}\|\cdot\|$, which we will call $\mathcal{T}_\infty$.

The problem with $\|\cdot\|$ is that it is not an algebra norm, as evidenced by the hexadecimalions. Furthermore, $(\mathbb{C}(\mathbf{R}),\mathcal{T})$ is not complete, its completion being $\ell^2(\mathbf{R})$ with its usual topology.

On the other hand, $(\mathbb{C}(\mathbf{R}),\mathcal{T}_\infty)$ is complete, and the product is (trivially) separately continuous ([N. Bourbaki (EVT)]), but it is not known if it is continuous.

For both topologies, any finite-dimensional vector space is closed and the restriction to that vector space is just the usual (euclidian) topology.

At any rate, given $x \in \mathbb{C}(\mathbf{R})$, the Powers Scholie proves that $\left(\mathrm{Vect}_{\mathbf{R}}(1,x), \|\ \|\right)$ is a commutative $\mathbf{R}$-Banach algebra (of dimension 1 if and only if $x \in \mathbf{R}$).

As a final thought, since $\mathbb{C}(\mathbf{A})$ is an $\mathbf{A}$-Cayley algebra, we can perform the Cayley doubling procedure on it! And again, and so on and so forth... We can actually perform an infinity of doubling as above, and embed all these doublings in what, essentially, is $\mathbf{A}[X,Y]$. And then we can start all over again... As we can readily see, there is no "ultimate" step... What seems to be going on is that we can build an object for any *finite* ordinal ([J.-M. Exbrayat, P. Mazet (1971)]), and we have built an object, which we have called $\mathbb{C}(\mathbf{A})$, for the first infinite ordinal ω. We have then seen that the doubling of $\mathbb{C}(\mathbf{A})$ yields the object corresponding to ω^* (the successor of ω). The next infinite ordinal with no predecessor (2ω) corresponds to $\mathbf{A}[X,Y]$. Further on (corresponding to ω^2), we find the set of polynomials in an indeterminate number of indeterminates (*i.e.*[11] $\mathbf{A}^{\left[\mathbf{N}^{[\mathbf{N}]}\right]}$). It is not clear, however, in which way we can extend the construction to any set of ordinals (*i.e.* there is no clear transfinite "recurrence formula").

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[11] Recal that if X is a monoïd ([N. Bourbaki (A)]) and Y is a set, $X^{[Y]}$ is the set of functions from Y to X which take values different from the neutral ement of X only for a finite numbers of elements of Y.

Chapter 3 The Exponential

1- What to find here

This chapter is mostly designed to prove the explicit formula for the exponential in $\mathbb{C}(\mathbf{R})$, and give several related results. As far as I known, these results are new.

There are many notions of the exponential, and many ways to see several of them. These, of course, agree when various different definitions can be put forward for the same object to be exponentiated. We will be concerned here mainly with the analyst's point of view, and define the exponential of quaternions thru the use of the usual power series ([A.F. Beardon (1979)],...). It is known that the approach detailed in [F. Pham (1996)] could also be used, at least for quaternions, though I believe it would then be necessary to derive the power series representation (or the intermediary differential representation we will also use) to achieve our present goal. It remains to be seen if it can also be carried over to the whole of $\mathbb{C}(\mathbf{R})$.

2- Definition

Given $x \in \mathbb{C}(\mathbf{R})$, we will call exponential of x, and we will write $\mathrm{Exp}(x)$ the element of $\mathbb{C}(\mathbf{R})$ given by $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$. The unambiguity and existence of $\mathrm{Exp}(x)$ is given by the fact that $\left(\mathrm{Vect}_{\mathbf{R}}(1,x), \|\ \|\right)$ is a commutative $\mathbf{R}$-Banach algebra, as we have said earlier. This, of course agrees with the definition on $\mathbf{R}$ and $\mathbf{C}$. We must bear in mind that $\mathrm{Exp}(x) \in \mathrm{Vect}_{\mathbf{R}}(1,x)$.

We see at once that $(\forall x \in \mathbb{C}(\mathbf{R}))\ \mathrm{Exp}(\bar{x}) = \overline{\mathrm{Exp}(x)}$. The exponential is continuous when restricted to each rung of $\mathbb{C}(\mathbf{R})$, and has its values into the same rung (we will give a more precise result later on).

3- Links with differentiation

Differentiating a function of one or several quaternions (or higher in the Cayley ladder) is quite problematic. Of course, since $\mathrm{Vect}_{\mathbf{R}}(1,x)$ is commutative, there is no ambiguity in defining $(\mathrm{f}(y)-\mathrm{f}(x))/(y-x)$ *if* $y \in \mathrm{Vect}_{\mathbf{R}}(1,x)$, and we can therefore differentiate $\mathrm{Exp}|_{\mathrm{Vect}_{\mathbf{R}}(1,x)}$ with respect to some $y \in \mathrm{Vect}_{\mathbf{R}}(1,x)$ and find that it is once again $\mathrm{Exp}|_{\mathrm{Vect}_{\mathbf{R}}(1,x)}$.

It is more fruitful, however, to differentiate a function of a *real* variable, with values in some topological $\mathbf{R}$-vector space.

Let us therefore consider, for some $x \in \mathbb{C}(\mathbf{R})$, the function $e_x : \left[\mathbf{R} \to \mathbb{C}(\mathbf{R}), t \mapsto \mathrm{Exp}(tx)\right]$. It is clear that e_x takes its values in $\mathrm{Vect}_{\mathbf{R}}(1,x)$, is differentiable and $e_x'(t) = x\, e_x(t) = e_x(t)\, x$, and of course $e_x(0) = 1$. This, of course proves that e_x is the unique solution to $f' = x f$, $f(0) = 1$ in $\mathcal{C}^1(\mathbf{R}, \mathrm{Vect}_{\mathbf{R}}(1,x))$, the set of one-time continuously differentiable functions from $\mathbf{R}$ to $\mathrm{Vect}_{\mathbf{R}}(1,x)$. Given any rung $\mathcal{E}$ of $\mathbb{C}(\mathbf{R})$ such that $x \in \mathcal{E}$, e_x is still the unique solution to $f' = x f$, $f(0) = 1$ in $\mathcal{C}^1(\mathbf{R}, \mathcal{E})$.

The perhaps surprising phenomenon is when we consider the equation $f' = x f$, $f(0) = \gamma$ in $\mathcal{C}^1(\mathbf{R}, \mathcal{E})$ for some rung $\mathcal{E}$ of $\mathbb{C}(\mathbf{R})$ such that $x \in \mathcal{E}$, and $\gamma \in \mathcal{E}$. If $\mathcal{E} = \mathbf{R}$ or $\mathcal{E} = \mathbf{C}$, then of course the solution is $e_x(t)\,\gamma$, and it turns out this is still true if $\mathcal{E} = \mathbf{H}$,

because of the associativity of the quaternionic product (this, actually, is how one can navigate the unit sphere of the quaternions, which is useful for interpolating between orientations, and was the problem under examination in [D. Pletincks (1989)]). It is interesting to note that this is *still* true if $\mathfrak{T}=\mathbf{O}$ because of the alternative nature of that algebra. This stops to be true with hexadecimalions, however. Indeed, consider $x=i+e'''$ and $\gamma=j$, and let $\mathrm{g}(t)=e_x(t)\,\gamma$. We will shortly see that $e_{i+e'''}\left(\frac{\pi\sqrt{2}}{4}\right)=\frac{\sqrt{2}}{2}(i+e''')$, from which we can deduce $\mathrm{g}\left(\frac{\pi\sqrt{2}}{4}\right)=\frac{\sqrt{2}}{2}(i+e''')j$ and $\mathrm{g}'\left(\frac{\pi\sqrt{2}}{4}\right)=\left((i+e''')\frac{\sqrt{2}}{2}(i+e''')\right)j=-\sqrt{2}j$ whereas $(i+e''')\mathrm{g}\left(\frac{\pi}{2}\right)=(i+e''')\left(\frac{\sqrt{2}}{2}(i+e''')j\right)=\sqrt{2}(-j+k''')$, and therefore $\mathrm{g}'\left(\frac{\pi}{2}\right)\neq(i+e''')\mathrm{g}\left(\frac{\pi}{2}\right)$. Numerical integration procedures will yield the solution to the differential equation, and therefore not the exponential function, unless care has been taken to chose the starting point correctly.

4- The closed formula for the exponential in $\mathbb{C}(\mathbf{R})$

We now give the main result of this work. Note that it is closed only in that we assume the exponential and classical trigonometric functions on $\mathbf{R}$ to be givens[12].

Theorem (Exponential):# If $x\in\mathbb{C}(\mathbf{R})$ then $\mathrm{Exp}(x)=\mathrm{e}^{\mathrm{Re}(x)}\left[\cos\left(\|\mathrm{Ur}(x)\|\right)+\mathrm{sinc}_\pi\left(\|\mathrm{Ur}(x)\|\right)\mathrm{Ur}(x)\right]$. $

Let $y\in\mathbb{C}(\mathbf{R})$ such that $\mathrm{Re}(y)=0$, $\|y\|=1$; then $y^2=\left[\mathcal{T}(y)-\bar{y}\right]y=-\mathcal{N}(y)=-1$. Therefore, in $\mathrm{Vect}_{\mathbf{R}}(1,y)$ computations are carried out exactly as in $\mathbf{C}$, with y taking the place of i. More precisely, $\left[\mathbf{C}\to\mathrm{Vect}_{\mathbf{R}}(1,y),a+ib\mapsto a+by\right]$ is a Banach isomorphism.

Let now $x\in\mathbb{C}(\mathbf{R})$. If $x\in\mathbf{R}$, we see the result is trivially true. Assume, then that $x\notin\mathbf{R}$, and let $\hat{x}=\frac{\mathrm{Ur}(x)}{\|\mathrm{Ur}(x)\|}$. Then $\mathrm{Re}(\hat{x})=0$, $\|\hat{x}\|=1$, $x=\mathrm{Re}(x)+\|\mathrm{Ur}(x)\|\hat{x}$, and of course $\mathrm{Vect}_{\mathbf{R}}(1,x)=\mathrm{Vect}_{\mathbf{R}}(1,\hat{x})$. The previous identification then allows us to find $\mathrm{Exp}(x)=\mathrm{e}^{\mathrm{Re}(x)}\left[\cos\left(\|\mathrm{Ur}(x)\|\right)+\sin\left(\|\mathrm{Ur}(x)\|\right)\hat{x}\right]$. $

As an example, we have, as announced earlier, $\mathrm{Exp}\left(\frac{\pi\sqrt{2}}{4}(i+e''')\right)=\frac{\sqrt{2}}{2}(i+e''')$.

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[12] A family of special functions will be of interest here, that of the "Sinus Cardinal" functions, defined for some parameter $a\in\mathbf{R}_+^*$ by $\mathrm{sinc}_a:\left[\mathbf{R}\to\mathbf{R},x\mapsto\frac{\sin\left(\frac{\pi x}{a}\right)}{\frac{x}{a}}\right]$. We will, by similitude, define the "Hyperbolic Sinus Cardinal" family of functions defined for some parameter $a\in\mathbf{R}_+^*$ by $\mathrm{sinhc}_a:\left[\mathbf{R}\to\mathbf{R},x\mapsto\frac{\sinh\left(\frac{\pi x}{a}\right)}{\frac{x}{a}}\right]$. These functions are entire functions on all of $\mathbf{R}$.

5- Some properties of the exponential and further consequences

We compute at once $\|\mathrm{Exp}(x)\| = \mathrm{e}^{|\mathrm{Re}(x)|}$.

As should be expected when we lose the benefit of commutativity, the exponential of a sum is in general different from the product of the exponentials; for instance we have $\mathrm{Exp}(i)\mathrm{Exp}(j) = \cos(1)\cos(1) + i\sin(1)\cos(1) + j\sin(1)\cos(1) + k\sin(1)\sin(1)$ whereas $\mathrm{Exp}(i+j) = \cos\left(\sqrt{2}\right) + \frac{\sqrt{2}}{2}(i+j)$. We also see immediately that the exponential is not injective on any rung $\mathcal{E}$ of $\mathbb{C}(\mathbf{R})$ containing $\mathbf{C}$, as it is already not injective on $\mathbf{C}$! We, however also lose the periodicity when $\mathcal{E}$ contains $\mathbf{H}$, as the periods would make an additive subgroup of $\mathcal{E}$ but the solutions of $\mathrm{Exp}(\mathrm{x}) = 1$ on $\mathbf{H}$ are exactly the set $2\pi.\mathbf{N}.\{0\} \times \mathbf{S}^2$ (with $\mathbf{S}^2$ the unit sphere of $\mathbf{R}^3$); the rest is number theory (and trying to fit square pegs into round holes). We have more details on the surjectivity of the exponential:

Corollary (sujectivity): # If $\mathcal{E}$ is a rung of $\mathbb{C}(\mathbf{R})$ containing $\mathbf{C}$, then the exponential is a surjection from $\mathcal{E}$ onto $\mathcal{E} - \{0\}$. $

We first note that given any $x \in \mathbb{C}(\mathbf{R})$, $\|\mathrm{Exp}(x)\| = \mathrm{e}^{|\mathrm{Re}(x)|}$ proves that the exponential never take the value 0 on $\mathbb{C}(\mathbf{R})$.

Let now $y \in \mathcal{E}$, $y \neq 0$. If $y \in \mathbf{R}$ we know we can solve our problem (in $\mathbf{R}$ if $y > 0$, in $\mathbf{C}$ if $y < 0$). Assume therefore that $y \notin \mathbf{R}$. We can find $\rho \in \mathbf{R}$ such that $\mathrm{e}^{\rho} = \|y\|$. Let $\tilde{y} = \frac{y}{\|y\|}$; $\|\tilde{y}\| = 1$ and $\tilde{y} \notin \mathbf{R}$, so let $\hat{y} = \frac{\mathrm{Ur}(\tilde{y})}{\|\mathrm{Ur}(\tilde{y})\|}$, so that $\tilde{y} = \mathrm{Re}(\tilde{y}) + \|\mathrm{Ur}(\tilde{y})\|\hat{y}$, $\mathrm{Re}(\tilde{y}) \neq 0$ and $\mathrm{Re}(\tilde{y})^2 + \|\mathrm{Ur}(\tilde{y})\|^2 = 1$. Let $\theta \in]0;\pi[$ the unique number such that $\cos(\theta) = \mathrm{Re}(\tilde{y})$ and $\sin(\theta) = \|\mathrm{Ur}(\tilde{y})\|$. We see that $\mathrm{Exp}(\rho + \theta\hat{y}) = y$. $

We can likewise find closed formulæ for other interesting entire functions (defining $\cos(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\sin(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cosh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$, $\sinh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$), to wit:

$$\cos(x) = \cos(\mathrm{Re}(x))\cosh(\|\mathrm{Ur}(x)\|) - \sin(\mathrm{Re}(x))\mathrm{sinhc}_{\pi}(\|\mathrm{Ur}(\mathrm{x})\|)\mathrm{Ur}(\mathrm{x})$$
$$\sin(x) = \sin(\mathrm{Re}(x))\cosh(\|\mathrm{Ur}(\mathrm{x})\|) + \cos(\mathrm{Re}(x))\mathrm{sinhc}_{\pi}(\|\mathrm{Ur}(\mathrm{x})\|)\mathrm{Ur}(\mathrm{x})$$
$$\cosh(x) = \cosh(\mathrm{Re}(x))\cos(\|\mathrm{Ur}(x)\|) + \sinh(\mathrm{Re}(x))\mathrm{sinc}_{\pi}(\|\ \|)\mathrm{Ur}(x)$$
$$\sinh(x) = \sinh(\mathrm{Re}(x))\cos(\|\mathrm{Ur}(x)\|) + \cosh(\mathrm{Re}(x))\mathrm{sinc}_{\pi}(\|\mathrm{Ur}(x)\|)\mathrm{Ur}(x)$$

and of course many other such.

6- Conclusion

We have found a closed formula for the exponential, for quaternions, octonions, and beyond.

An interesting application of this formula is navigation on the unit sphere of the quaternions, leading to an algorithm for the interpolation of orientations, but which, in general, does not preserve the horizontal. This can also be achieved, however, and has been implemented by the author and a colleague ([Horizontal-preserving quaternions]).

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